Algorithms for Generating Convex Sets in Acyclic Digraphs

P. Balister* S. Gerke† G. Gutin‡ A. Johnstone §
J. Reddington¶ E. Scott‖ A. Soleimanfallah** A. Yeo††

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Abstract

A set X of vertices of an acyclic digraph D is convex if $X \neq \emptyset$ and there is no directed path between vertices of X which contains a vertex not in X. A set X is connected if $X \neq \emptyset$ and the underlying undirected graph of the subgraph of D induced by X is connected. Connected convex sets and convex sets of acyclic digraphs are of interest in the area of modern embedded processor technology. We construct an algorithm $\mathcal{A}$ for enumeration of all connected convex sets of an acyclic digraph D of order n. The time complexity of $\mathcal{A}$ is $O(n \cdot cc(D))$, where $cc(D)$ is the number of connected convex sets in D. We also give an optimal algorithm for enumeration of all (not just connected) convex sets

*Department of Mathematical Sciences, University of Memphis, TN 38152-3240, USA. E-mail: pbalistr@memphis.edu

†Department of Mathematics, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: stefanie.gerke@rhul.ac.uk

‡Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: gutin@cs.rhul.ac.uk

§Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: adrian@cs.rhul.ac.uk

¶Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: joseph@cs.rhul.ac.uk

‖Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: eas@cs.rhul.ac.uk

**Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: arezou@cs.rhul.ac.uk

††Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK, E-mail: anders@cs.rhul.ac.uk

of an acyclic digraph D of order n. In computational experiments we demonstrate that our algorithms outperform the best algorithms in the literature.

Using the same approach as for $\mathcal{A}$, we design an algorithm for generating all connected sets of a connected undirected graph G. The complexity of the algorithm is $O(n \cdot c(G))$, where n is the order of G and $c(G)$ is the number of connected sets of G. The previously reported algorithm for connected set enumeration is of running time $O(mn \cdot c(G))$, where m is the number of edges in G.

1 Introduction

A set X of vertices of an acyclic digraph D is *convex* if $X \neq \emptyset$ and there is no directed path between vertices of X which contains a vertex not in X. A set X is *connected* if $X \neq \emptyset$ and the underlying undirected graph of the subgraph of D induced by X is connected. A set is *connected convex* (a *cc-set*) if it is both connected and convex.

In Section 3, we introduce and study an algorithm $\mathcal{A}$ for generating all connected convex sets of a connected acyclic digraph D of order n. The running time of $\mathcal{A}$ is $O(n \cdot cc(D))$, where $cc(D)$ is the number of connected convex sets in D. Thus, the algorithm is (almost) optimal with respect to its time complexity. Interestingly, to generate only k cc-sets using $\mathcal{A}$ we need $O(n^3 + kn)$ time. In Section 5, we give experimental results demonstrating that the algorithm is practical on reasonably large data dependency graphs for basic blocks generated from target code produced by Trimaran [22] and SimpleScalar [3]. Our experiments show that $\mathcal{A}$ is better than the state-of-the-art algorithm of Chen, Maskell and Sun [6]. Moreover, unlike the algorithm in [6], our algorithm has a provable (almost) optimal worst time complexity.

Although such algorithms are of less importance in our application area because of wider scheduling issues, there also exist algorithms that enumerate all of the convex sets of an acyclic graph. Until recently the algorithm of choice for this problem was that of Atasu, Pozzi and Ienne [2, 19], however the CMS algorithm [6] (run in general mode) outperforms the API algorithm in most cases. In Section 4, we give a different algorithm, for enumeration of all the convex sets of an acyclic digraph, which significantly outperforms the CMS and API algorithms and which has a (optimal) runtime performance of the order of the sum of the sizes of the convex sets.

Avis and Fukuda [4] designed an algorithm for generating all connected sets in a connected graph G of order n and size m with time complexity $O(mn \cdot c(G))$ and space complexity $O(n+m)$, where $c(G)$ is the number of connected sets in G. Observe that when G is bipartite there is an orientation D of G such that every connected set of G corresponds to a cc-set of D and vice versa. To obtain D orient every edge of G from X to Y, where X and Y are the partition classes of G.

The algorithm of Avis and Fukuda is based on a so-called reverse search. Applying the approach used to design the algorithm $\mathcal{A}$ to connected set enumeration, in Section 6, we describe an algorithm $\mathcal{C}$ for generating all connected sets in a connected graph G of order n with much better time complexity, $O(n \cdot c(G))$. This demonstrates that our approach can be applied with success to various vertex set/subgraph enumeration problems. The space complexity of our algorithm matches that of the algorithm of Avis and Fukuda.

1.1 Algorithms Applications

There is an immediate application for $\mathcal{A}$ in the field of so-called *custom computing* in which central processor architectures are parameterized for particular applications.

An embedded or *application specific* computing system only ever executes a single application. Examples include automobile engine management systems, satellite and aerospace control systems and the signal processing parts of mobile cellular phones. Significant improvements in the price-performance ratio of such systems can be achieved if the instruction set of the application specific processor is specifically tuned to the application.

This approach has become practical because many modern integrated circuit implementations are based on Field Programmable Gate Arrays (FPGA). An FPGA comprises an array of logic elements and a programmable routing system, which allows detailed design of logic interconnection to be performed directly by the customer, rather than a complete (and very high cost) custom integrated circuit having to be produced for each application. In extreme cases, the internal logic of the FPGA can even be modified whilst in operation.

Suppliers of embedded processor architectures are now delivering *extensible*

versions of their general purpose processors. Examples include the ARM OptimoDE [1], the MIPS Pro Series [18] and the Tensilica Xtensa [21]. The intention is that these architectures be implemented either as traditional logic with an accompanying FPGA containing the hardware for extension instructions, or be completely implemented within a large FPGA. By this means, hardware development has achieved a new level of flexibility, but sophisticated design tools are required to exploit its potential.

The goal of such tools is the identification of time critical or commonly occurring patterns of computation that could be directly implemented in custom hardware, giving both faster execution and reduced program size, because a sequence of base machine instructions is being replaced by a single custom *extension* instruction. For example, a program solving simultaneous linear equations may find it useful to have a single instruction to perform matrix inversion on a set of values held in registers.

The approach proceeds by first locating the *basic blocks* of the program, regions of sequential computation with no control transfers into them. For each basic block we construct a *data dependency graph* (DDG) which contains vertices for each base (unextended) instruction in the block, along with a vertex for each initial input datum. Figure 1 shows an example of a DDG. There is an arc to the vertex for the instruction u from each vertex whose instruction computes an input operand of u. DDG's are acyclic because execution within a basic block is by definition sequential.

Extension instructions are combinations of base machine instructions and are represented by sets of the DDG. In Figure 1, sections A and B are convex sets that represent candidate extension instructions. However, Section B is not connected. If such a region were implemented as a single extension instruction we should have separate independent hardware units within the instruction. Although this presents no special difficulties, and in Section 4 we give an optimal algorithm for constructing all such sets, present engineering practice is to restrict the search to connected convex components on the grounds that unconnected convex components are composed of connected ones, and that the system's code scheduler will perform better if it is allowed to arrange the independent computations in different ways at different points in the program.

Unlike connectivity however, convexity is not optional. An extension instruction cannot perform computations that depend on instructions external to the extension instruction. This means that there can be no data flows

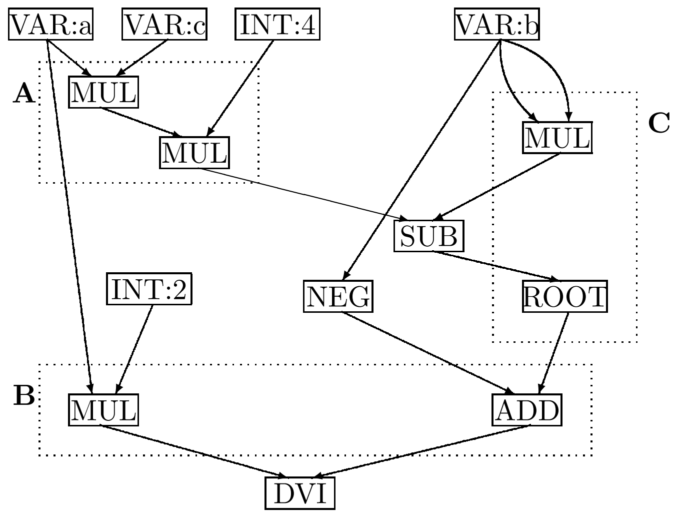


Figure 1: Data dependency graph for $\frac{-b+\sqrt{b^2-4ac}}{2a}$

out of and then back into the extension instruction: the set corresponding to an extension instruction must be convex. Thus section C in Figure 1 does not represent a candidate extension instruction since it breaches the 'no external computation rule' because it is non-convex: there is a path *via* the `SUB` node that is not in the set.

Ideally we would like to fully consider all possible candidate instructions and select the combination which results in the most efficient implementation. In practice this is unlikely to be feasible as, in worst case, the number of candidates will be exponential in the number of original program instructions. However, it is useful to have a process which can find all the potential instructions, even if the set of instructions used for final consideration has to be restricted. In this work we only deal with generation of a set of possible candidate instructions. Interested readers can refer to [19, 25].

1.2 Related Theoretical Research

Many other algorithms for special vertex set/subgraph generation have been studied in the literature. Kreher and Stinson [16] describe an algorithm for generating all cliques in a graph G of order n with running time $O(n \cdot cl(G))$, where $cl(G)$ is the number of cliques in G.

Several algorithms have been suggested for the generation of all spanning trees in a connected graph G of order n and size m. Let t be the number of spanning trees in G. The first spanning trees generating algorithms [11, 17, 20] used backtracking which is useful for enumerating various kinds of subgraphs such as paths and cycles. Using the algorithms from [17, 20], Gabow and Myers [11] suggested an algorithm with time complexity $O(tn + n + m)$ and space complexity $O(n + m)$. If we output all spanning trees by their edges, this algorithm is optimal in terms of time and space complexities. Later algorithms of a different type were developed; these algorithms (see, e.g., [15, 23, 24]) find a new spanning tree by exchanging a pair of edges. As a result, the algorithms of Kapoor and Ramesh [15] and Shioura and Tamura [23] require only $O(t + n + m)$ time and $O(nm)$ space. The algorithm of Shioura, Tamura and Uno [24] is of the same optimal running time, but also of optimal space: $O(n + m)$.

An out-tree is an orientation of a tree such that all vertices but one are of in-degree 1. Kapoor, Kumar and Ramesh [14] presented an algorithm for enumerating all spanning out-trees of a digraph with n vertices, m arcs and t spanning out-trees. The algorithm takes $O(\log n)$ time per spanning tree; more precisely, it runs in $O(t \log n + n^2\alpha(n, n) + nm)$, where α is the Inverse Ackermann function. It first outputs a single spanning out-tree and then a list of arc swaps; each spanning out-tree can be generated from the first spanning out-tree by applying a prefix of this sequence of arc swaps.

2 Terminology, Notation and Preliminaries

Let D be a digraph. If xy is an arc of D ($xy \in A(D)$), we say that y is an *out-neighbor* of x and x is an *in-neighbor* of y. The set of out-neighbors of x is denoted by $N_D^+(x)$ and the set of in-neighbors of x is denoted by $N_D^-(x)$. For a set X of vertices of D, its *out-neighborhood* (resp. *in-neighborhood*) is $N_D^+(X) = \bigcup_{x \in X} N_D^+(x) \setminus X$ (resp. $N_D^-(X) = \bigcup_{x \in X} N_D^-(x) \setminus X$). A digraph D^{TC} is called the *transitive closure* of D if $V(D^{TC}) = V(D)$ and a vertex x is an in-neighbor of a vertex y in D^{TC} if and only if there is a path from x to y in D.

Let S be a non-empty set of vertices of a digraph D. A directed path P of D is an *S-path* if P has at least three vertices, its initial and terminal vertices are in S and the rest of the vertices are not in S. For a digraph D, $\mathcal{CC}(D)$ ($\mathcal{CO}(D)$) denotes the collection of cc-sets (convex sets) in D;

cc$(D) = |\mathcal{CC}(D)|$ and co$(D) = |\mathcal{CO}(D)|$. An ordering $v_1, v_2, \ldots, v_n$ of vertices of an acyclic digraph D is called *acyclic* if for every arc v_iv_j of D we have $i < j$.

Lemma 2.1. *Let D be a connected acyclic digraph and let S be a vertex set in D. Then S is a cc-set in D if and only if it is a cc-set in D^{TC}.*

Proof. Let S be a set of vertices of D. We will first prove that there is an S-path in D if and only if there is an S-path in D^{TC}. Since all arcs of D are in D^{TC}, every S-path in D is an S-path in D^{TC}. Let $Q = x_1x_2 \ldots x_q$ be an S-path in D^{TC}. Then there are paths $P_2, P_3, \ldots, P_q$ such that $Q' = x_1P_2x_2P_3x_3 \ldots x_{q-1}P_qx_q$ is a path in D (Q' must be a path since D is acyclic). Since x_1 and x_q belong to S and x_2 does not belong to S, there is a subpath of Q' which is an S-path.

If S is connected in D then it is clearly connected in D^{TC}, which implies that if S is a cc-set in D then it is a cc-set in D^{TC}. Now let S be a cc-set in D^{TC}. Assume that $D[S]$ is not connected and let x and y be vertices in different connected components in $D[S]$, but which are connected by an arc in D^{TC}. Without loss of generality xy is the arc in D^{TC} and Q is a path from x to y in D. However as S is convex all vertices in Q also belong to S and therefore x and y belong to the same connected component in $D[S]$, a contradiction. □

It is well-known (see, e.g., the paper [9] by Fisher and Meyer, or [10] by Furman) that the transitive closure problem and the matrix multiplication problem are closely related: there exists an $O(n^a)$-algorithm, with $a \geq 2$, to compute the transitive closure of a digraph of order n if and only if the product of two boolean $n \times n$ matrices can be computed in $O(n^a)$ time. Coppersmith and Winograd [7] showed that there exists an $O(n^{2.376})$-algorithm for the matrix multiplication. Thus, we have the following:

Theorem 2.2. *The transitive closure of a digraph of order n can be found in $O(n^{2.376})$ time.*

We will need the following two results proved in [12].

Theorem 2.3. *For every connected acyclic digraph D of order n, $cc(D) \geq n(n+1)/2$. If an acyclic digraph D of order n has a Hamiltonian path, then $cc(D) = n(n+1)/2$.*

Theorem 2.4. *Let $f(n) = 2^n + n + 1 - d_n$, where $d_n = 2 \cdot 2^{n/2}$ for every even n and $d_n = 3 \cdot 2^{(n-1)/2}$ for every odd n. For every connected acyclic digraph D of order n, $cc(D) \leq f(n)$. Let $\vec{K}_{p,q}$ denote the digraph obtained from the complete bipartite graph $K_{p,q}$ by orienting every edge from the partite set of cardinality p to the partite set of cardinality q. We have $\mathrm{cc}(\vec{K}_{a,n-a}) = f(n)$ provided $|n - 2a| \leq 1$.*

3 Algorithm for Generating CC-Sets of an Acyclic Digraph

In this section D denotes a connected acyclic digraph of order n and size m. Now we describe the main algorithm of this paper; we denote it by $\mathcal{A}$. The input of $\mathcal{A}$ is D and $\mathcal{A}$ outputs all cc-sets of D. The formal description of $\mathcal{A}$ is followed by an example and proofs of correctness of $\mathcal{A}$ and its complexity. Finally, we show that to produce k cc-sets $\mathcal{A}$ requires $O(n^{2.376} + kn)$ time. The algorithm works as follows. Given a digraph D on n vertices, it considers an acyclic ordering $v_1, \ldots, v_n$ of the transitive closure of D. For each vertex v_i we consider the sets $X = \{v_i\}$ and $Y = \{v_{i+1}, \ldots, v_n\}$ and call the subroutine $\mathcal{B}(X, Y, D)$ which finds all cc-sets S in D such that $X \subseteq S \subseteq X \cup Y$. At each step, if possible $\mathcal{B}(X, Y, D)$ removes an element v from Y and adds it to X. If X has out-neighbors we choose v to be the 'largest' out-neighbor in the acyclic ordering(line 3), otherwise if X has in-neighbors we choose v to be the 'smallest' in-neighbor (line 8). Then we find the other vertices required to maintain convexity (line 4 or line 9). If there are no in- or out-neighbors we output X, otherwise we find find all the cc-sets such that $X \subseteq S \subseteq X \cup Y$ and $v \in S$ (line 12) and then all the cc-sets such that $X \subseteq S \subseteq X \cup Y$ and $v \notin S$ (line 13).

Step 1: Find the transitive closure of D and set $D = D^{TC}$.

Step 2: Find an acyclic ordering $v_1, v_2, \ldots, v_n$ of D.

Step 3: For each $i = 1, 2, \ldots, n$ do the following. Set $X := \{v_i\}$, $Y := \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ and call $\mathcal{B}(X, Y, D)$.

Step 4 subroutine $\mathcal{B}(X, Y, D)$:

1. set $A = N^{+}_{D^{TC}}(X) \cap Y$

2. **if** $A \neq \emptyset$ {

3. set $v = v_j$, where j =max$\{i : \; v_i \in A\}$

4. set $R = \{v\} \cup (N^-_{D^{TC}}(v) \cap A)$ }

5. **else** {

6. set $B = N^-_{D^{TC}}(X) \cap Y$

7. **if** $B \neq \emptyset$ {

8. set $v = v_k$, where k =min$\{i : \; v_i \in B\}$

9. set $R = \{v\} \cup (N^+_{D^{TC}}(v) \cap B)$ } }

10. **if** $A = \emptyset$ and $B = \emptyset$ { output X }

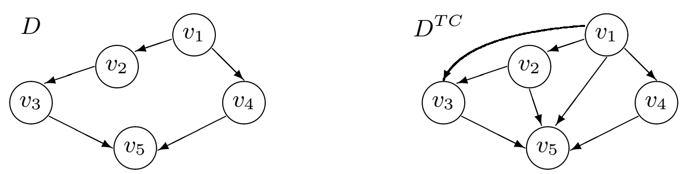
11. **else** {

12. $\mathcal{B}(X \cup R,\; Y \backslash R,\; D)$

13. $\mathcal{B}(X,\; Y \backslash \{v\},\; D)$ } }

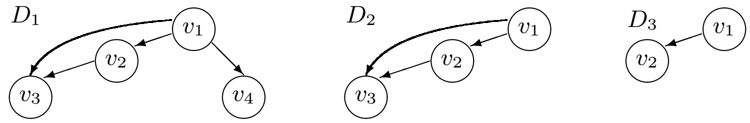
Before proving the correctness of $\mathcal{A}$, we consider an example.

Example 3.1. *Let D be the graph on the left below*



In Step 1, we find $A(D^{TC}) = A(D) \cup \{v_1v_3, v_2v_5, v_1v_5\}$ (above right). Observe that v_1, v_2, v_3, v_4, v_5 is an acyclic ordering. We may assume that this is the ordering found in Step 2.

For $i = 1$ in Step 3, we have $X = \{v_1\}$ and $Y = \{v_2, v_3, v_4, v_5\} = N^+(X)$, and we call $\mathcal{B}(\{v_1\}, \{v_2, v_3, v_4, v_5\}, D)$. Then in Step 4, line 1, we compute $A = \{v_2, v_3, v_4, v_5\}$ and then, lines 3 and 4, obtain $v = v_5$, $N^-_{D^{TC}}(v) = \{v_1, v_2, v_3, v_4\}$ and $R = \{v_2, v_3, v_4, v_5\}$. Then, at line 12, we make a recursive call to $\mathcal{B}(V(D), \emptyset, D)$. In this call we have $A = B = \emptyset$ so, at line 10, the set $V(D) = \{v_1, \ldots, v_5\}$ is output and the recursive call returns, to line 13 of $\mathcal{B}(\{v_1\}, \{v_2, v_3, v_4, v_5\}, D)$, where we make a call to $\mathcal{B}(\{v_1\}, \{v_2, v_3, v_4\}, D)$. We are now effectively looking at the graph D_1 below.



In Step 4, lines 1-4, we compute $A = \{v_2, v_3, v_4\}$ *and obtain* $v = v_4$, $N^-_{DTC}(v) = \{v_1\}$ *and* $R = \{v_4\}$. *At lines 12 and 13 we make recursive calls to* $\mathcal{B}(\{v_1, v_4\}, \{v_2, v_3\}, D)$ *and* $\mathcal{B}(\{v_1\}, \{v_2, v_3\}, D)$ *respectively.*

In the call to $\mathcal{B}(\{v_1, v_4\}, \{v_2, v_3\}, D)$*, lines 1-4, we obtain* $v = v_3$ *and* $R = \{v_2, v_3\}$. *This in turn generates calls to* $\mathcal{B}(\{v_1, v_4, v_2, v_3\}, \emptyset, D)$*, which just outputs* $\{v_1, v_2, v_3, v_4\}$ *and returns, and* $\mathcal{B}(\{v_1, v_4\}, \{v_2\}, D)$. *The latter call generates calls to* $\mathcal{B}(\{v_1, v_4, v_2\}, \emptyset, D)$ *and* $\mathcal{B}(\{v_1, v_4\}, \emptyset, D)$*, which output* $\{v_1, v_2, v_4\}$ *and* $\{v_1, v_4\}$*, respectively.*

In the call to $\mathcal{B}(\{v_1\}, \{v_2, v_3\}, D)$*, where we are effectively looking at* D_2 *above, we obtain* $v = v_3$ *and* $R = \{v_2, v_3\}$. *This in turn generates calls to* $\mathcal{B}(\{v_1, v_2, v_3\}, \emptyset, D)$*, which just outputs* $\{v_1, v_2, v_3\}$ *and returns, and* $\mathcal{B}(\{v_1\}, \{v_2\}, D)$ *(graph* D_3 *above). The latter call generates calls to* $\mathcal{B}(\{v_1, v_2\}, \emptyset, D)$ *and* $\mathcal{B}(\{v_1\}, \emptyset, D)$*, which output* $\{v_1, v_2\}$ *and* $\{v_1\}$*, respectively. This completes the case* $i = 1$ *in Step 3, and all the cc-sets containing* v_1 *have been output.*

Now we perform Step 3 with $i = 2$*, effectively looking at the graph* D_4.

The call to $\mathcal{B}(\{v_2\}, \{v_3, v_4, v_5\}, D)$ *generates further recursive calls in the following order*

- $\mathcal{B}(\{v_2, v_5, v_3\}, \{v_4\}, D)$
  - $\mathcal{B}(\{v_2, v_5, v_3, v_4\}, \emptyset, D)$*, output* $\{v_2, v_3, v_4, v_5\}$
  - $\mathcal{B}(\{v_2, v_5, v_3\}, \emptyset, D)$*, output* $\{v_2, v_3, v_5\}$
- $\mathcal{B}(\{v_2\}, \{v_3, v_4\}, D)$
  - $\mathcal{B}(\{v_2, v_3\}, \{v_4\}, D)$*, output* $\{v_2, v_3\}$
  - $\mathcal{B}(\{v_2\}, \{v_4\}, D)$*, output* $\{v_2\}$.

Thus all the cc-sets containing v_2 *but not* v_1 *are output.*

Performing Step 3 again with $i = 3$*, effectively looking at the graph* D_5 *above, the call to* $\mathcal{B}(\{v_3\}, \{v_4, v_5\}, D)$*, generates the following recursive calls*

$\mathcal{B}(\{v_3, v_5\}, \{v_4\}, D)$

$\mathcal{B}(\{v_3, v_5, v_4\}, \emptyset, D)$, *output* $\{v_3, v_4, v_5\}$

$\mathcal{B}(\{v_3, v_5\}, \emptyset, D)$, *output* $\{v_3, v_5\}$

$\mathcal{B}(\{v_3\}, \{v_4\}, D)$, *output* $\{v_3\}$

which ouput all the cc-sets containing v_3 *but not* v_1 *or* v_2*.*

For the case $i = 4$ *in Step 3 we get the following calls*

$\mathcal{B}(\{v_4\}, \{v_5\}, D)$

$\mathcal{B}(\{v_4, v_5\}, \emptyset, D)$, *output* $\{v_4, v_5\}$

$\mathcal{B}(\{v_4\}, \emptyset, D)$, *output* $\{v_4\}$

and for $i = 5$ *we get*

$\mathcal{B}(\{v_5\}, \emptyset, D)$, *output* $\{v_5\}$

after which $\mathcal{A}$ *terminates.*

Lemma 3.2. *Algorithm* $\mathcal{A}$ *correctly outputs all cc-sets of* D*.*

Proof. Recall, the convex (connected) sets of D are precisely the convex (connected) sets of D^{TC}. We prove the result for D^{TC}.

Firstly we show that all the sets X output by $\mathcal{A}$ are in $\mathcal{CC}(D^{TC})$. We will show that within $\mathcal{A}$, for any call $\mathcal{B}(X, Y, D)$ we have that $X \cap Y = \emptyset$, $X \cup Y$ is convex and X is a cc-set. This is clearly sufficient as X is the only set output.

These properties hold for Step 3 when $\mathcal{B}(\{v_i\}, \{v_{i+1}, \dots, v_n\}, D)$ is called as we have chosen an acyclic ordering of the vertices. Thus we assume that the properties hold for the sets X, Y and consider the pairs of sets $X \cup R$, $Y \backslash R$ and X, $Y \backslash \{v\}$ constructed in $\mathcal{B}(X, Y, D)$. In both cases clearly the intersections are empty, and since $R \subseteq N^+_{D^{TC}}(X) \cup N^-_{D^{TC}}(X)$, $X \cup R$ is connected.

Now we will prove that $X \cup R$ is convex. Suppose that there is a path u, y, w where $u, w \in X \cup R$. Note that if there exists an $(X \cup R)$-path then by transitivity of D^{TC} there exists an $(X \cup R)$-path of length two. By convexity of $X \cup Y$ we have $y \in X \cup Y$. Also, $y \neq v$ as we have chosen v to be either the maximal element of $N^+_{D^{TC}}(X)$ or the minimal element of $N^-_{D^{TC}}(X)$, and D^{TC} is transitive and thus the presence of the arcs uy and yw implies the presence of the arc uw. Assume that $A \neq \emptyset$. Then $R \subseteq N^+_{D^{TC}}(X)$. Since $u \in X \cup N^+_{D^{TC}}(X)$ and the arc uy exists, the transitivity of D^{TC} implies that $y \in N^+_{D^{TC}}(X)$. Since X is convex it follows that not both vertices u, w can

be in X and that there is no arc from $N^+_{D^{TC}}(X)$ to X. Thus $w \notin X$ and so $w \in R \subseteq N^-_{D^{TC}}(X)$. By the transitivity of D^{TC} and the fact that yw exists and that $w \in N^-_{D^{TC}}(X)$ we have $y \in N^-_{D^{TC}}(X)$ and thus $y \in R$. Similarly if $A = \emptyset$ then $R \subseteq N^-_{D^{TC}}(X)$ and by the transitivity of D^{TC} and since $w \in X \cup N^-_{D^{TC}}(X)$ we have $u \in R$ and thus $y \in N^-_{D^{TC}}(X) \cap N^+D^{TC}(X)$.

Secondly we show that if $X \neq \emptyset$ is cc then X is output by $\mathcal{A}$. If S is a cc-set and $j = \min\{i : \ v_i \in S\}$ then $\{v_j\} \subseteq S \subseteq \{v_j, v_{j+1}, \dots, v_n\}$. Thus it is sufficient to show that if S is cc and $X \subseteq S \subseteq X \cup Y$ then $\mathcal{B}(X, Y, D)$ outputs S. We prove this by induction on $|Y|$.

If $(N^+_{D^{TC}}(X) \cap Y) = \emptyset = (N^-_{D^{TC}}(X) \cap Y)$ then, since S is connected, $S = X$ and $\mathcal{B}(X, Y, D)$ outputs X at line 10. This proves the result for $|Y| = 0$, and for $|Y| \geq 1$ we may assume that $v \in (N^+_{D^{TC}}(X) \cup Y \cup N^-_{D^{TC}}(X))$.

If $v \notin S$ then we have $X \subseteq S \subseteq (X \cup (Y \backslash \{v\}))$ and $|Y \backslash \{v\}| < |Y|$, so by induction the call to $\mathcal{B}(X, Y \backslash \{v\}, D)$ at line 13 outputs S. If $r \in (R \backslash \{v\})$, we have arcs rv and xr, for some $x \in X \subseteq S$. Thus, if $v \in S$, by convexity of S we have $R \subseteq S$. Then, since $|Y \backslash R| < |Y|$, the call to $\mathcal{B}(X \cup R, Y \backslash R, D)$ at line 12 outputs S. ☐

Lemma 3.3. *The running time of $\mathcal{A}$ is $O(n \cdot cc(D))$.*

Proof. Note that by Theorem 2.3 and the fact that D is connected we have $n \times cc(D) \geq n^2(n + 1)/2$. Therefore the transitive closure of D can be found in $O(n \cdot cc(D))$ time, by Theorem 2.2. It is well-known that an acyclic ordering can be found in time $O(n + m)$, see, e.g., [5], and clearly the sets $N^+_{D^{TC}}(v)$ and $N^-_{D^{TC}}(v)$ can be computed at the start of the algorithm in $O(n)$ time, for each $v \in V(D)$.

We will now show that $\mathcal{B}(X, Y, D)$ runs in time $O(|Y| \cdot cc'(X, Y) + K_{X,Y})$, where $cc'(X, Y)$ is the number of cc-sets S such that $X \subseteq S \subseteq X \cup Y$ and $K_{X,Y}$ is the sum of the sizes of the sets S. Note that $\mathcal{B}$ returns at line 10 or makes two recursive calls to $\mathcal{B}$ (lines 12,13). If $\mathcal{B}$ returns at line 10 then we call this a *leaf* call otherwise the function call is an *internal* call. All function calls can be viewed as nodes of a binary tree (every node is a leaf or has two children) whose leaves and internal nodes correspond to calls to $\mathcal{B}$. It is easy to see, by induction, that the number of internal nodes equals the number of leaves minus one. It is easy to see, by induction on the depth of the call tree, that $\mathcal{B}$ outputs each set S only once ($\mathcal{B}(X \cup R, Y \backslash R, D)$ and $\mathcal{B}(X, Y \backslash \{v\}, D)$

output those that contain v and do not contain v, respectively). Thus we have $cc'(X,Y)$ leaf calls and $cc'(X,Y)-1$ internal calls.

We assume that the set implementation allows us to find the size of a set and the largest and smallest elements of the set in unit time. Then the time taken by a call $\mathcal{B}(X,Y,D)$ depends on the time taken to calculate the sets A, B and R. Since $A,B \subseteq Y$, the time to compute R is at most $O(|Y|)$. If we implement $\mathcal{B}(X,Y,D)\cap Y$ so that $N^+_{D^{TC}}(X)\cap Y$ and $N^-_{D^{TC}}(X)$ are passed in as parameters then the time taken to calculate A and B is at most $O(|Y|)$. By definition of R we have that $N^+_{D^{TC}}(X\cup R) = N^+_{D^{TC}}(X) - R$ and $N^-_{D^{TC}}(X\cup R) = N^-_{D^{TC}}(X)\cup N^-_{D^{TC}}(v) - R - X$ provided $A \neq \emptyset$, and $N^-_{D^{TC}}(X\cup R) = N^-_{D^{TC}}(X) - R$ and $N^+_{D^{TC}}(X\cup R) = N^+_{D^{TC}}(X)\cup N^+_{D^{TC}}(v) - R - X$ provided $A = \emptyset$ (and $B \neq \emptyset$). Since $R \subseteq Y$, these sets can be computed in $O(|Y|)$ time.

If $\mathcal{B}(X,Y,D)$ calls $\mathcal{B}(X',Y',D)$ then $|Y'| < |Y|$ thus a call to $\mathcal{B}$ at an internal node takes at most $O(|Y|)$ time, and a call at a leaf node takes at most $O(|Y|+|X|)$ time, giving the desired total time bound of $O(|Y|\cdot cc'(X,Y)+K_{X,Y})$.

We let K_i denote the sum of the sizes of all the cc-sets S such that $v_i \in S \subseteq \{v_{i+1},\ldots,v_n\}$, and observe that $K_1+\ldots+K_n \leq n\cdot cc(D)$.

Finally, by Step 3, we conclude that the total running time is

$$O\left(\sum_{i=1}^{n} cc'(\{v_i\},\{v_{i+1},v_{i+2},\ldots,v_n\})\cdot(n-i)+K_i\right) = O(cc(D)\cdot n).$$

$\square$

Theorem 3.4. *Algorithm $\mathcal{A}$ is correct and its time and space complexities are $O(n\cdot cc(D))$ and $O(n^2)$, respectively.*

Proof. The correctness and time complexity follows from the two lemmas above. The space complexity is dominated by the space complexity of Step 1, $O(n^2)$. $\square$

Since $cc(D)$ may well be exponential, we may wish to generate only a restricted number k of cc-sets. Theorem 3.5 can be viewed as a result in fixed-parameter algorithmics [8] with k being a parameter.

Theorem 3.5. *To output k cc-sets the algorithm $\mathcal{A}$ requires $O(n^{2.376} + kn)$ time.*

Proof. We may assume that k is at most the number of cc-sets containing vertex v_1 since otherwise the proof is analogous.

We consider the binary tree T introduced in the proof of Lemma 3.3 and prove our claim by induction on k. It takes $O(n^{2.376})$ time to perform Steps 1,2 and 3. It takes $O(n)$ internal nodes of T to reach the first leaf of T and, thus, for $k = 1$ we obtain $O(n^{2.376} + n)$ time. Assume that $k \geq 2$. Let x be the first leaf of T reached by $\mathcal{A}$, let y be the parent of x on T, let z be another child of y on T and let u be the parent of y. Observe that after deleting the nodes x and y and adding an edge between u and z, we obtain a new binary tree T'. By induction hypothesis, to reach the first $k-1$ leaves in T', we need $O(n^{2.376} + (k-1)n)$ time. To reach the first k leaves in T, we need to reach x and the first $k-1$ leaves in T'. Thus, we need to add to $O(n^{2.376} + (k-1)n)$ the time required to visit x and y only, which is $O(n)$. Thus, we have proved the desired bound $O(n^{2.376} + kn)$. $\square$

4 Generating Convex Sets in Acyclic Digraphs

It is not hard to modify $\mathcal{A}$ such that the new algorithm will generate all convex sets of an acyclic digraph D in time $O(n \cdot co(D))$, where $co(D)$ is the number of convex sets in D. However, a faster algorithm is possible and we present one in this section.

To obtain all convex sets of D (and $\emptyset$, which is not convex by definition), we call the following recursive procedure with the original digraph D and with $F = \emptyset$. This call yields an algorithm whose properties are studied below.

A vertex x is a *source* (*sink*) if it has no in-neighbors (out-neighbors). In general, the procedure $\mathcal{CS}$ takes as input an acyclic digraph $D = (V, A)$ and a set $F \subseteq V$ and outputs all convex sets of D which contain F. The procedure $\mathcal{CS}$ outputs V and then considers all sources and sinks of the graph that are not in F. For each such source or sink s, we call $\mathcal{CS}(D - s, F)$ and then add s to F. Thus, for each sink or source $s \in V \setminus F$ we consider all sets that contain s and all sets that do not contain s.

$\mathcal{CS}(D = (V, A), F)$

1. **output** V

2. **for all** $s \in V \setminus F$ with $|N^+(s)| = 0$ or $|N^-(s)| = 0$ **do** $\{$

3. **for all** vertices v find $N^+_{D-s}(v)$ and $N^-_{D-s}(v)$

4. call $\mathcal{CS}(D - s, F)$; set $F := F \cup \{s\}$

5. **for all** vertices v find $N^+_D(v)$ and $N^-_D(v)$ $\}$

4.1 Correctness of the procedure

Proposition 4.2 and Theorem 4.3 imply that the procedure $\mathcal{CS}$ is correct. We first show that all sets generated in line 1 are, in fact, convex sets. To this end, we use the following lemma.

Lemma 4.1. *Let D be an acyclic graph, let X be a convex set of D, and let $s \in X$ be a source or sink of $D[X]$. Then $X \setminus \{s\}$ is a convex set of D.*

Proof. Suppose that $X \setminus \{s\}$ is not convex in D. Then there exist two vertices $u, v \in X \setminus \{s\}$ and a directed path P from u to v which contains a vertex not in $X \setminus \{s\}$. Since X is convex, P only uses vertices of X and in particular $s \in P$. Thus, there is a subpath $u'sv'$ of P with $u', v' \in X$. But since s is a source or a sink in $D[X]$ such a subpath cannot exist, a contradiction. $\square$

Now we can prove the following proposition.

Proposition 4.2. *Let $D = (V, A)$ be an acyclic digraph and let $F \subseteq V$. Then every set output by $\mathcal{CS}(D, F)$ is convex.*

Proof. We prove the result by induction on the number of vertices of the outputted set. The entire vertex set V is convex and is outputted by the procedure. Now assume all sets of size $n - i \geq 2$ that are outputted by the procedure are convex. We will show that all sets of size $n - i - 1$ that are outputted are also convex. When a set C is outputted the procedure $\mathcal{CS}(D[C], F')$ was called for some set $F' \subseteq V$. The only way $\mathcal{CS}(D[C], F')$ can be invoked is that there exist a set $C' \subset V$ and a source or sink c of $D[C']$ with $C = C' \setminus \{c\}$. Moreover C' will be outputted by the procedure and, thus, by our assumption is convex. The result now follows from Lemma 4.1. $\square$

Theorem 4.3. *Let $D = (V, A)$ be an acyclic digraph and let $F \subseteq V$. Then every convex set of D containing F is outputted exactly once by $\mathcal{CS}(D, F)$.*

Proof. Let C be a convex set of D containing F. We first claim that there exist vertices $c_1, c_2, \ldots, c_t \in V$ with $V = \{c_1, c_2, \ldots, c_t\} \cup C$ and c_i is a source or sink of $D[C \cup \{c_i, c_{i+1}, \ldots, c_t\}]$ for all $i \in \{1, 2, \ldots, t\}$. To prove the claim we will show that for every convex set H with $C \subset H \subseteq V$, there exists a source or sink $s \in H \setminus C$ of the digraph $D[H]$. This will prove our claim as by Lemma 4.1 $H \setminus \{s\}$ is a convex set of D and we can repeatedly apply the claim.

If there exists no arc from a vertex of C to a vertex of $D[H \setminus C]$ then any source of $H \setminus C$ is a source of $D[H]$. Note that $D[H \setminus C]$ is an acyclic digraph and, thus, has at least one source (and sink). Thus we may assume that there is an arc from a vertex u of C to a vertex v of $H \setminus C$. Consider a longest path $v = v_1 v_2 \ldots v_r$ in $D[H \setminus C]$ leaving v. Observe that v_r is a sink of $D[H \setminus C]$ and, moreover, there is no arc from v_r to any vertex of C since otherwise there would be a directed path from $u \in C$ to a vertex in C containing vertices in $H \setminus C$ which is impossible as C is convex. Hence v_r is a sink of $D[H]$ and the claim is shown.

Next note that a sink or source remains a sink or source when vertices are deleted. Thus when $\mathcal{CS}(D, F)$ is executed and a source or sink s is considered, then we distinguish the cases when $s = c_i$ for some $i \in \{1, 2, \ldots, t\}$ or when this is not the case. If $s = c_i$ and we currently consider the digraph D' and the fixed set F', then we follow the execution path calling $\mathcal{CS}(D' - s, F')$. Otherwise we follow the execution path that adds s to the fixed set. When the last c_i is deleted, we call $\mathcal{CS}(D[C], F'')$ for some F'' and the set C is outputed. It remains to show that there is a unique execution path yielding C. To see this, note that when we consider a source or sink s then either it is deleted of moved to the fixed set F. Thus every vertex is considered at most once and then deleted or fixed. Therefore each time we consider a source or sink there is a unique decision that finally yields C. $\square$

4.2 Running time of $\mathcal{CS}$

We assume that the input acyclic digraph $D = (V, A)$ is given by the two adjacency lists for each vertex, and the number of in-neighbors and out-neighbors is stored for each vertex. One can obtain this information at the

beginning in $O(n+m)$ time, where n (m) is the number of vertices (arcs) of the input connected acyclic digraph D. Observe that we output the vertex set of D as one convex set. Thus, it suffices to show that the running time of $\mathcal{CS}(D,F)$ without the recursive calls is $O(|V|)$. This will yield the running time $O(\sum_{C\in\mathcal{CO}(D)}|C|)$ of $\mathcal{CS}$ by Theorem 4.3.

Since we have stored the number of in-neighbors and out-neighbors for every vertex $v \in V$, we can determine *all* sources and sinks in $O(|V|)$ time. For the recursive calls of $\mathcal{CS}$ we delete one vertex and have to update the number of in- respectively out-neighbors of all neighbors of the deleted vertex s. The vertex s has at most $|V|-1$ neighbors and we can charge the cost of the updating information to the call of $\mathcal{CS}(D-s,F)$. Moreover we store the neighbours of s so that we can reintroduce them after the call of $\mathcal{CS}(D-s,F)$. Moving the sinks and sources to F needs constant time for each source or sink and thus we obtain $O(|V|)$ time in total.

In summary we initially need $O(n+m)$ time, and then each call of $\mathcal{CS}(D,F)$ is charged with $O(|V|)$ before it is called and then additionally with $O(|V|)$ time during its execution. Since we output a convex set of size $O(|V|)$, the total running time is $O(n+m)+O(\sum_{C\in\mathcal{CO}(D)}|C|)$. Since $\sum_{C\in\mathcal{CO}(D)}|C| = \Omega(n^2)$ by Theorem 2.3, the running time of $\mathcal{CS}$ is $O(\sum_{C\in\mathcal{CO}(D)}|C|)$.

5 Implementation and Experimental Results

In order to test our algorithms $\mathcal{A}$ and $\mathcal{CS}$ for practicality we have implemented and run them on several instances of DDG's of basic blocks. We have compared our algorithm with the state-of-the-art algorithm of Chen, Maskell and Sun [6] (the CMS algorithm) using their own implementation, but with the code for I/O constraint checking removed so as to ensure that their algorithm was not disadvantaged. For completeness we have also compared $\mathcal{CS}$ to Atasu, Pozzi and Ienne's algorithm [19] (the API06 algorithm). All the algorithms were coded in C++ and all experiments were carried out on a 2 x Dual Core AMD Opteron 265 1.8GHz processor with 4 Gb RAM, running SUSE Linux 10.2 (64 bit).

Our first set of tests is based on C and C++ programs taken from the benchmark suites of MiBench [13] and Trimaran [22]. We compiled these benchmarks for both the Trimaran (A,B,C,D,E) and SimpleScalar [3] (F,G,H,I) architectures. From here we examined the control-flow graph for each pro-

ID	NV	NA	NS	CMS (CT)	$\mathcal{A}$ (CT)
A	35	38	139,190	170	96
B	42	45	4,484,110	5,546	3,246
C	26	28	5,891	6	4
D	39	94	3,968,036	4,346	2,710
E	45	44	1,466,961	1,750	1,156
F	24	22	46,694	60	30
G	20	19	397	0	0
H	20	21	1,916	0	0
I	43	47	10,329,762	13,146	7,210

Table 1: cc-sets for benchmark programs

gram to select a basic block within a critical loop of the program (often this block had been unrolled to some degree to increase the potential for efficiency improvements).

We considered basic blocks, ranging from 20 to 45 lines of low level, intermediate, code, for which we generated the DDGs. We then selected, from these DDGs, the non-trivial connected components on which to run our algorithms.

We give some preference to benchmarks which suite the intended application of the research taking our test cases from security applications including benchmarks for the Advanced Encryption Standard (B,C) and safety-critical software (A, E). We also include a basic example from the Trimaran benchmark suite: Hyper (D), an algorithm that performs quick sort (F), part of a jpeg algorithm (G), and an example from the fft benchmark in mibench containing C source code for performing Discrete Fast Fourier Transforms (H). The final example is taken from the standard blowfish benchmark, an encryption algorithm.

The results we have obtained are given in Table 1. In the following tables NV denotes the number of vertices, NS denotes the number of generated sets, NA number of arcs, CT denotes clock time in 10^{-3} CPU seconds, and for the benchmark data ID identifies the benchmark.

For examples G and H both algorithms ran in almost 0 time. For the other examples, the above results demonstrate that our algorithm $\mathcal{A}$ outperforms the CMS algorithm.

NV	NA	NS	CMS (CT)	$\mathcal{A}$ (CT)
15	56	32,400	30	16
16	64	65,041	56	23
17	72	130,322	114	60
18	81	261,139	240	113
19	90	522,722	540	253
20	100	1,046,549	1,080	513
21	110	2,094,102	2,166	1,048
22	121	4,190,231	4,086	2,156

Table 2: cc-sets for graphs with maximum number of cc-sets

ID	NV	NA	NS	API06	CMS (CT)	$\mathcal{CS}$ (CT)
A	35	38	1,123,851	2,560	1,390	270
C	26	28	120,411	250	120	40
F	24	22	3,782,820	3,250	3,630	860
G	20	19	122,111	70	120	30
H	20	21	55,083	110	110	20

Table 3: All convex sets for benchmark programs

We also consider examples with worst-case numbers of cc-sets. Let, as in Theorem 2.4, $\vec{K}_{p,q}$ denote the digraph obtained from the complete bipartite graph $K_{p,q}$ by orienting every edge from the partite set of cardinality p to the partite set of cardinality q. By Theorem 2.4 the digraphs $\vec{K}_{a,n-a}$ with $|n-2a| \leq 1$ have the maximum possible number of cc-sets. Our experimental results for digraphs $\vec{K}_{a,n-a}$ with $|n-2a| \leq 1$ are given in Table 2. Again we see that $\mathcal{A}$ outperforms the CMS algorithm.

We have compared algorithm $\mathcal{CS}$ with both CMS running in 'unconnected' mode and with API06. The examples used are the same as in Table 1, however we do not give results for examples B, D, E and I as these graphs produce an extremely large number of convex sets and as a result, do not terminate in reasonable time. The results are shown in Table 3. We can see that although CMS generally out-performs API06, there are two cases where API06 is marginally better. However, $\mathcal{CS}$ is consistantly three to five times faster than either of the other algorithms.

NV	NA	NS	API06	CMS (CT)	$\mathcal{CS}$ (CT)
15	56	32,768	40	40	10
16	64	65,536	70	70	30
17	72	131,072	140	130	60
18	81	261,144	320	320	130
19	90	524,288	720	700	320
20	100	1,046,575	1,590	1,500	710
21	110	2,097,152	3,320	3,010	1,500
22	121	4,194,304	7,140	6,310	3,120

Table 4: All convex sets for graphs with maximum number of cc-sets

For interest we have also compared API06, CMS and $\mathcal{CS}$ on the digraphs that have maximal numbers of cc-sets. The results are shown in Table 4. Again, while CMS and API06 are roughly comparable, $\mathcal{CS}$ is a least twice as fast as both of them.

6 Connected Sets Generation Algorithm

Let G be a connected (undirected) graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let G have m edges. For a vertex $x \in V(G)$ and a set $X \subseteq V(G)$, let $N(x) = \{z \in V(G) : \ xz \in E(G)\}$ and $N(X) = \bigcup_{x \in X} N(x) \setminus X$. The following is an algorithm, $\mathcal{C}$, for generating all connected sets of G.

Step 1: For each $i = 1, 2, \ldots, n$ do the following. Set $X := \{v_i\}$ and $Y := \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. Initiate the set N_X as $N_X := N(X) \cap Y$.

Step 2 (subroutine $\mathcal{D}$): *Comment: $\mathcal{D}$ finds all connected sets Q in D such that $X \subseteq Q \subseteq X \cup Y$.*

(2a): If $N_X = \emptyset$ then return the connected set X (and stop).

(2b): If $N_X \neq \emptyset$, then let $v \in N_X$ be arbitrary.

(2c): *Comment: In this step we will find all connected sets S such that $X \cup \{v\} \subseteq S \subseteq (X \cup Y)$.*
Set $N_{X,0} := N_X$, $X_0 := X$ and $Y_0 := Y$. Remove v from Y and N_X, and add it to X. For every $u \in Y \setminus N_X$ check whether u has an edge to v and if it does then add it to N_X.

Make a recursive call to subroutine $\mathcal{D}$. *Comment: we consider the new X and Y.*

Change N_X, X, and Y back to their original state by setting $N_X := N_{X,0}$, $X := X_0$, and $Y := Y_0$.

(2d): *Comment: In this step we will find all connected sets S such that $X \subseteq S \subseteq (X \cup Y)$ and $v \notin S$.* Remove v from Y and remove v from N_X.

Make a recursive call to subroutine $\mathcal{D}$.

Change Y back to its original state by adding v back to Y. Also change N_X back to its original state by adding v to it.

Similarly to Theorem 3.4, one can prove the following:

Theorem 6.1. *Let $c(G)$ be the number of connected sets of a connected graph G. Algorithm $\mathcal{C}$ is correct and its time and space complexities are $O(n \cdot c(G))$ and $O(n + m)$, respectively.*

7 Discussions and Open Problems

Our computational experiments show that $\mathcal{A}$ performs well and is of definite practical interest. We have tried various heuristic approaches to speed up the algorithm in practice, but all approaches were beneficial for some instances and inferior to the original algorithm for some other instances. Moreover, no approach could significantly change the running time. The algorithm was developed independently from the CMS algorithm. However, the two algorithms are closely related, and work continues to isolate the implementation effects that give the performance differences.

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